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On the linearization of Regge calculus

Snorre H. Christiansen*

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Abstract

We study the linearization of three dimensional Regge calculus around Euclidean metric. We provide an explicit formula for the corresponding quadratic form and relate it to the curl T curl operator which appears in the quadratic part of the Einstein-Hilbert action and also in the linear elasticity complex. We insert Regge metrics in a discrete version of this complex, equipped with densily defined and commuting interpolators. We show that the eigenpairs of the curl T curl operator, approximated using the quadratic part of the Regge action on Regge metrics, converge to their continuous counterparts, interpreting the computation as a non-conforming finite element method.

1 Introduction

Regge calculus [31] is an approach to general relativity [39] which to some extent predates the explosion of numerical methods in the natural sciences. Spacetime is represented by a simplicial complex. Given this simplicial complex a finite dimensional space of metrics is defined. We call such metrics Regge metrics. A functional defined on this space of metrics and mimicking the Einstein-Hilbert action, is provided. We call this functional the Regge action. A critical point of the Regge action on the space of Regge metrics is generally believed to be a good approximation to a true solution of Einstein's equations of general relativity [26].

Regge calculus is quite popular in studies of quantum gravity [32]. Its discrete nature also makes it a natural candidate for the construction of efficient algorithms to simulate the classical field equations, a possibility expressed already in Regge's paper. In this direction we are aware of [6] and [20]. However it seems that the bulk of numerical relativity computations are performed using other methods [25]. In particular recent progress on the simulation of merging black holes [29] appear to be carried out in sophisticated but in some sense traditional finite difference schemes. We hope that this paper can contribute to establishing Regge calculus as a good alternative. Its geometric "coordinate free" nature would make it a structure-preserving method in the sense of [21] and could be decisive in particular for simulating cosmologies, where global qualitative effects are important.

*CMA, University of Oslo, PO Box 1053 Blindern, NO-0316 Oslo, Norway. email : snorrec@math.uio.no

We are not aware of any stringent convergence results for Regge calculus, except those of [14]. There it is shown that the curvature of the Regge metrics interpolating a smooth metric, converges in the sense of measures when the mesh width goes to 0. In [16] we related the space of Regge metrics to Whitney forms. As remarked in [9], Whitney forms correspond to lowest order mixed finite elements [30][28] for which one has a relatively well developed convergence theory [11][34]. We showed that there is a natural basis for the space of Regge metrics expressed in terms of Whitney forms and that second order differential operators restricted to Courant elements are in one to one correspondence with linear forms on Regge metrics, edge elements and Courant elements. This link integrates Regge calculus into the finite element framework. However we did not approach the question of curvature which is central to Regge calculus.

In this paper we further develop the theory of Regge elements. We insert them in a complex of spaces equipped with densily defined interpolators providing commuting diagrams as in finite element exterior calculus [3]. More importantly we provide results concerning the Regge action.

A priori it is not clear if Regge calculus should be considered a conforming or a non-conforming finite element method. Is the Regge action the restriction to Regge metrics of some extension by continuity of the Einstein-Hilbert action, to a large enough class of non-smooth metrics? One might compare with Wilson's lattice gauge theory discretization of the Yang-Mills equations [35], where the discrete action is *not* a simple restriction of the continuous one.

Given a metric, the scalar curvature multiplied with the volume form provided by the metric, is a certain density on spacetime depending on the metric in a highly non-linear way. For a piecewise smooth metric such as a Regge metric, partial derivatives should be defined in the sense of distributions. The process of associating a distribution with a function depends on an L^2 product which in turn depends on the metric. For a non-smooth metric partial derivatives of arbitrary order are not defined, even in the sense of distributions. This problem comes in addition to the fact that products of distributions are not defined in general. It is not clear that, on Regge metrics, the ill-defined partial derivatives and non-linearities appearing in the Einstein-Hilbert action, cancel in such a way that the only thing remaining is the Regge action. The arguments put forward by Regge to justify the definition of the Regge action are entirely integral in nature, bearing on the holonomy groups, given by the parallel transport along paths avoiding the codimension-2 skeleton of the simplicial complex, where the curvature sits.

If we expand the Einstein-Hilbert action in small perturbations around Minkowski spacetime, the linear term is 0 since the Minkowski metric solves the Einstein equations. The first non-trivial term is a quadratic form $\mathcal{Q}$ whose spatial part is associated with the $\mathrm{curl}\,\mathrm{T}\,\mathrm{curl}$ operator appearing in the linear elasticity complex [2]. We show that this (spatial) quadratic form, defined a priori for smooth fields, has a natural extension to (spatial) Regge metrics. Moreover we show that this extension corresponds to the quadratic part of the Regge action, establishing that the first non-trivial terms of the Regge action and the Einstein-Hilbert action agree. However the natural Hilbert space on which the quadratic form is continuous (just) fails to contain the Regge elements! We argue that Regge calculus is a minimally non-conforming method.

The Euler-Lagrange equations corresponding to finding critical points of $\mathcal{Q}$, on "all" metrics, are nothing but the linearized Einstein equations. They govern the propagation of gravitational waves on flat spacetime, so they have some interest by themselves. One might wish to simulate them using some variant of linearized Regge calculus. One difficulty is the gauge freedom which imposes constraints on the evolution. Outside the constraint manifold, the evolution equation is no longer hyperbolic. We are still working on how to best tackle this problem but as a step to analyse the convergence of numerical methods based on Regge calculus we show that the eigenvalues for the $\mathrm{curl}\,\mathrm{T}\,\mathrm{curl}$ operator are well approximated on Regge elements. The problem of hyperbolicity is that there are eigenvalues of arbitrary magnitude of both signs; one of the signs corresponds to modes that are excluded by the constraints in the continuous case. The theory we develop is inspired by works on the eigenvalue problem for Maxwell's equations [23][8][13]. As for Maxwell's equations the operator does not have a compact resolvent (due to the existence of an infinite dimensional kernel), so the basic theory [5] has to be amended. Indeed it has been shown that in this situation, stability is not sufficient to get eigenvalue convergence [7]. Additional difficulties are the above mentioned lack of sign, whereby Cea type arguments have to be replaced by inf-sup conditions [4][10], as well as the (limit) non-conformity of the method. Central to the argument is an analogue for metrics of the Hodge decomposition of differential forms.

The paper is organized as follows. In section 2 we study Regge elements in finite element terms. In section 3 we relate the linearized Regge action to the $\mathrm{curl}\,\mathrm{T}\,\mathrm{curl}$ operator. In section 4 we study the discrete eigenvalue problem.

2 Regge elements

Let U be a domain in $\mathbb{R}^3$ partitioned into tetrahedra by a simplicial complex $\mathcal{T}_h$. The parameter h denotes the largest diameter of a simplex of $\mathcal{T}_h$. For the most part we shall neglect boundary conditions. Indeed I don't know how they are treated in Regge calculus. Thus we shall work with periodic boundary conditions.

Regge metrics are symmetric matrix fields on U that are piecewise constant with respect to $\mathcal{T}_h$ and such that for any two tetrahedra sharing a triangle as a face, the tangential-tangential component of the metric is continuous across the face. In other words the pullback of the metric, seen now as a bilinear form, to the interface is the same from both sides. This space of metrics has one degree of freedom per edge, essentially the length of the edge provided by the metric. We proceed to give basic properties of this space, as in finite element exterior calculus.

In [16] we related this space of metrics, which we denote by X_h, to Whitney forms. For each vertex $x \in \mathcal{T}_h^0$ let λ_x denote the corresponding barycentric coordinate map. Then the following family of metrics is a basis of X_h, indexed by edges $i \in \mathcal{T}_h^1$:

$$1/2(\mathrm{d}\lambda_{x_i} \otimes \mathrm{d}\lambda_{y_i} + \mathrm{d}\lambda_{y_i} \otimes \mathrm{d}\lambda_{x_i}), \tag{1}$$

where the vertices of the edge i are denoted x_i and y_i.

Degrees of freedom can be defined as follows. For any edge $i \in \mathcal{T}_h^1$ consider the linear form on continuous metrics:

$$u \mapsto \int_0^1 u_{x_i+s(y_i-x_i)}(y_i - x_i, y_i - x_i)\mathrm{d}s. \tag{2}$$

One checks that on a tetrahedron these degrees of freedom are unisolvent on the constant metrics. They also guarantee tangential-tangential continuity.

Let I_h denote the corresponding interpolator onto X_h. It is defined on $\mathrm{H}^{1+\delta}(U)$ for $\delta > 0$ (functions in $\mathrm{H}^{1+\delta}(U)$ have, in three dimensions, well-defined L^2 traces on edges) and we have an error estimate:

$$|u - I_h u| \leq Ch\|u\|_{\mathrm{H}^{1+\delta}}, \tag{3}$$

where $|\cdot|$ denotes L^2 norm.

Recall that the $\operatorname{curl} \mathrm{T} \operatorname{curl}$ operator is defined on symmetric matrix fields by taking first the curl of its columns then transposing and then taking once again the curl of its columns. It yields a symmetric matrix field. We derive an expression for $\operatorname{curl} \mathrm{T} \operatorname{curl} u$ when $u \in X_h^1$.

Lemma 2.1. *Let T be an portion of a plane in $\mathbb{R}^3$ with a piecewise smooth boundary. Let n be the oriented normal. The inward pointing normal to ∂T is denoted m. Let v be a vector, considered as a constant possibly non-tangential, vectorfield on T. Let δ_T denote the Dirac surface measure on T, δ'_T the double layer distribution (so that $\operatorname{grad} \delta_T = n\delta'_T$) and $\delta_{\partial T}$ the Dirac line measure on ∂T. Regard $v\delta_T$ as a distribution on $\mathbb{R}^3$. Then we have, in the sense of distributions:*

$$\operatorname{curl}(v\delta_T) = (v \times n)\delta'_T - (v \times m)\delta_{\partial T}. \tag{4}$$

Proof. The oriented tangent on ∂T is denoted t. Thus (m, n, t) is an oriented orthonormal basis of $\mathbb{R}^3$. By definition we have for any smooth compactly supported vector field ϕ on $\mathbb{R}^3$:

$$\langle \operatorname{curl}(v\delta_T), \phi \rangle = \langle v\delta_T, \operatorname{curl} \phi \rangle = \int_T v \cdot \operatorname{curl} \phi. \tag{5}$$

We have:

$$\operatorname{curl} \phi = \partial_n(\phi \times n) + \operatorname{div}_T(\phi \times n)n + \operatorname{grad}_T(\phi \cdot n) \times n. \tag{6}$$

The first term of the right hand side of this equation gives:

$$\int_T v \cdot \partial_n(\phi \times n) = -\int_T (v \times n) \cdot \partial_n \phi, \tag{7}$$

which by definition is the first term of the right hand side of (4).

The second term of the right hand side of (6) gives:

$$\int_T v \cdot \operatorname{div}_T(\phi \times n)n = -\int_{\partial T} (v \cdot n)(\phi \times n) \cdot m = \int_{\partial T} (v \cdot n)(\phi \cdot t), \tag{8}$$

whereas the third term gives:

$$\int_T v \cdot \operatorname{grad}_T(\phi \cdot n) \times n = \int_{\partial T} (v \times n)(\phi \cdot n) \cdot m = -\int_{\partial T} (v \cdot t)(\phi \cdot n). \tag{9}$$

These two terms combine into:

$$-\int_{\partial T} (v \times m) \cdot \phi, \tag{10}$$

which corresponds to the second term of the right hand side of (4). $\square$

For any vector v, let $\operatorname{skew} v$ be the skew symmetric matrix defined by:

$$(\operatorname{skew} v)v' = v' \times v. \tag{11}$$

Lemma 2.2. *Consider a sector between two halfplanes P_0 and P_1 originating from a common edge with unit tangent t. The unit outwardpointing normal on the planes is denoted n. The inward pointing normals to the edge in the planes are denoted m_0 and m_1. We let $n_i = \pm n$ be the normal to P_i such that (m_i, n_i, t) is oriented. Upon relabelling we may suppose $n_0 = -n$ and $n_1 = n$. The Dirac surface measure on the boundary of the sector is denoted δ and the double-layer distribution δ'. The Dirac line measure on the edge is denoted δ_e. In the sector consider a constant metric u, extended by 0 outside of it. Then:*

$$\operatorname{curl} \mathrm{T} \operatorname{curl} u = -\,(\operatorname{skew} n)u(\operatorname{skew} n)\delta' \tag{12}$$

$$+\,((\operatorname{skew} m_1)u(\operatorname{skew} n_1) - (\operatorname{skew} m_0)u(\operatorname{skew} n_0))\delta_e. \tag{13}$$

Proof. We have:

$$\operatorname{curl} u = (\operatorname{skew} n)u\delta. \tag{14}$$

Hence:

$$\mathrm{T} \operatorname{curl} u = -u(\operatorname{skew} n)\delta. \tag{15}$$

We now apply the preceding lemma to the columns of this matrix and get the result. ☐

Remark 2.1. It is not clear from the above expression that $\operatorname{curl} \mathrm{T} \operatorname{curl}$ is a symmetric matrix field. However if we denote by R_ρ the rotation around the vector t by an angle ρ, there is an angle θ such that R_θ sends the basis (m_0, n_0, t) to (m_1, n_1, t). Put:

$$A(\rho) = (\operatorname{skew} R_\rho m_0)u(\operatorname{skew} R_\rho n_0), \tag{16}$$

so that:

$$A(\theta) - A(0) = (\operatorname{skew} m_1)u(\operatorname{skew} n_1) - (\operatorname{skew} m_0)u(\operatorname{skew} n_0). \tag{17}$$

Remark that taking derivatives with respect to θ gives:

$$A'(\rho) = (\operatorname{skew} R_\rho n_0)u(\operatorname{skew} R_\rho n_0) - (\operatorname{skew} R_\rho n_0)u(\operatorname{skew} R_\rho n_0), \tag{18}$$

which is symmetric. Therefore its integral must be symmetric.

For any edge i, let δ_i denote the Dirac line measure on i and t_i the unit oriented tangent vector along i. Let j be a face having i as an edge. We let m_{ij} be the unit vector in the face j, orthogonal to the edge i and pointing into the face. We let n_{ij} be the unit vector orthogonal to the face j oriented such that (m_{ij}, n_{ij}, t_i) is an oriented basis of $\mathbb{R}^3$. The vector n_{ij} depends on i only for its orientation. Let $[u]_{ij}$ be the jump of u across face j in the order of n_{ij}.

Proposition 2.3. *We have:*

$$\operatorname{curl} \mathrm{T} \operatorname{curl} u = \sum_i [u]_i t_i t_i^{\mathrm{T}} \delta_i, \tag{19}$$

where we sum over edges i and:

$$[u]_i = \sum_j n_{ij}^{\mathrm{T}} [u]_{ij} m_{ij}, \tag{20}$$

where we sum over faces j containing the edge i.

Proof. We use the preceding lemma. The double layer distributions cancel two by two, by the tangential-tangential continuity of u. We have:

$$\operatorname{curl} \mathrm{T} \operatorname{curl} u = \sum_i \sum_j (\operatorname{skew} m_{ij})[u]_{ij}(\operatorname{skew} n_{ij})\delta_i. \tag{21}$$

Remark that the columns of the matrix $[u]_{ij}(\operatorname{skew} n_{ij})$ are proportional to n_{ij} (by tangential-tangential continuity of u). In other words there is a vector α_{ij} such that:

$$[u]_{ij}(\operatorname{skew} n_{ij}) = n_{ij}\alpha_{ij}^{\mathrm{T}}. \tag{22}$$

Then write:

$$(\operatorname{skew} m_{ij})[u]_{ij}(\operatorname{skew} n_{ij}) = (\operatorname{skew} m_{ij})n_{ij}\alpha_{ij}^{\mathrm{T}} = -t_i\alpha_{ij}^{\mathrm{T}}. \tag{23}$$

This matrix has columns proportional to t_i. It follows that the colums of $\sum_j (\operatorname{skew} m_{ij})[u]_{ij}(\operatorname{skew} n_{ij})$ must be proportional to t_i. Since in addition this matrix is symmetric, it can be written:

$$\sum_j (\operatorname{skew} m_{ij})[u]_{ij}(\operatorname{skew} n_{ij}) = s_i t_i t_i^{\mathrm{T}}. \tag{24}$$

The scalar coefficient s_i is determined by taking traces:

$$\begin{aligned} s_i &= \operatorname{tr} \sum_j (\operatorname{skew} m_{ij})[u]_{ij}(\operatorname{skew} n_{ij}) && (25)\\ &= -\operatorname{tr} \sum_j (t_i n_{ij}^{\mathrm{T}} - n_{ij}t_i^{\mathrm{T}})[u]_{ij}(\operatorname{skew} n_{ij}) && (26)\\ &= -\sum_j \operatorname{tr}(n_{ij}^{\mathrm{T}}[u]_{ij}(\operatorname{skew} n_{ij})t_i) - \operatorname{tr}(t_i^{\mathrm{T}}[u]_{ij}(\operatorname{skew} n_{ij})n_{ij}) && (27)\\ &= \sum_j n_{ij}^{\mathrm{T}}[u]_{ij}m_{ij}, && (28) \end{aligned}$$

as announced. □

The space space $X_h = X_h^1$ can be inserted in a complex of spaces X_h^k with $0 \leq k \leq 3$, each equipped with a densily defined interpolator I_h^k.

We let X_h^0 denote the space of continuous piecewise affine vector fields, equipped with the nodal interpolator I_h^0. Recall that I_h^0 is defined on $\mathrm{H}^{3/2+\delta}$ for $\delta > 0$. The deformation operator (the symmetric gradient), denoted def, induces a map $X_h^0 \to X_h^1$.

We let X_h^2 denote the space of matrix edge measures of the form:

$$u = \sum_{i \in \mathcal{T}_h^1} u_i t_i t_i^{\mathrm{T}} \delta_i \tag{29}$$

where for each edge i, u_i is a real number, t_i is the unit oriented tangent vector to i and δ_i is the Dirac line measure on i. The preceding proposition shows that $\operatorname{curl} \mathrm{T} \operatorname{curl}$ induces a map $X_h^1 \to X_h^2$. The standard L^2 duality extends to a non-degenerate bilinear form on $X_h^2 \times X_h^1$.

We define an interpolator I_h^2 onto X_h^2 by requiring:

$$\forall v \in X_h^1 \quad \langle I_h^2 u, v\rangle = \langle u, v\rangle. \tag{30}$$

We remark that for all $u \in \mathrm{L}^2$ and all $v \in \mathrm{H}^{1+\delta}$:

$$\langle I_h^2 u, v\rangle = \langle I_h^2 u, I_h^1 v\rangle = \langle u, I_h^1 v\rangle. \tag{31}$$

From (3) we deduce, for $\delta > 0$:

$$\|u - I_h^2 u\|_{\mathrm{H}^{-1-\delta}} \leq Ch|u|. \tag{32}$$

We let X_h^3 denote the space of vector vertex measures of the form:

$$u = \sum_{i \in \mathcal{T}_h^0} u_i \delta_i, \tag{33}$$

where for each vertex i, u_i is a vector and δ_i is the Dirac measure attached to i. The standard L^2 duality extends to a non-degenerate bilinear form on $X_h^3 \times X_h^0$. We define an interpolator I_h^3 onto X_h^3 by requiring:

$$\forall v \in X_h^0 \quad \langle I_h^3 u, v\rangle = \langle u, v\rangle. \tag{34}$$

As in the preceding case we remark that for all $u \in \mathrm{L}^2$ and all $v \in \mathrm{H}^{3/2+\delta}$:

$$\langle I_h^3 u, v\rangle = \langle I_h^3 u, I_h^0 v\rangle = \langle u, I_h^0 v\rangle. \tag{35}$$

Let $\mathbb{V}$ denote the space of vectors, and $\mathbb{S}$ denote the space of symmetric matrices. We let C^∞ denote the space of smooth real functions. The space of smooth vector fields (resp. symmetric matrix fields) can be identified with $C^\infty \otimes \mathbb{V}$ (resp. $C^\infty \otimes \mathbb{S}$).

Proposition 2.4. *We have a commuting diagram of spaces:*

$$\begin{tikzcd}
C^\infty \otimes \mathbb{V} \arrow[r, "\mathrm{def}"] \arrow[d, "I_h^0"] & C^\infty \otimes \mathbb{S} \arrow[r, "\mathrm{curl}\,\mathrm{T}\,\mathrm{curl}"] \arrow[d, "I_h^1"] & C^\infty \otimes \mathbb{S} \arrow[r, "\mathrm{div}"] \arrow[d, "I_h^3"] & C^\infty \otimes \mathbb{V} \arrow[d, "I_h^3"] \\
X_h^0 \arrow[r, "\mathrm{def}"] & X_h^1 \arrow[r, "\mathrm{curl}\,\mathrm{T}\,\mathrm{curl}"] & X_h^2 \arrow[r, "\mathrm{div}"] & X_h^3
\end{tikzcd} \tag{36}$$

On the lower row the linear operators are defined in the sense of distributions.

Proof. For any $u \in C^\infty \otimes \mathbb{S}$ and any $v \in X_h^1$, we have:

$$\langle I_h^2 \operatorname{curl} \mathrm{T} \operatorname{curl} u, v\rangle = \langle \operatorname{curl} \mathrm{T} \operatorname{curl} u, v\rangle \tag{37}$$

$$= \langle u, \operatorname{curl} \mathrm{T} \operatorname{curl} v\rangle \tag{38}$$

$$= \langle I_h^1 u, \operatorname{curl} \mathrm{T} \operatorname{curl} v\rangle \tag{39}$$

$$= \langle \operatorname{curl} \mathrm{T} \operatorname{curl} I_h^1 u, v\rangle. \tag{40}$$

As a consequence we get commutation of the middle square.

Commutation of the first square is straightforward and commutation of the last square follows by duality. □

3 Linearizing the Regge action

We are concerned with the three-dimensional case.

We consider first what happens around a single edge. Fix an oriented line (edge) in $\mathbb{R}^3$ with unit tangent t. Originating from this edge are half-planes (faces) indexed by a cyclic parameter i and ordered counter-clockwise. The sector between edges i and $i+1$ is indexed by $i+1/2$. Let m_i be the oriented unit length vector in half-plane i which is orthogonal to t. Let n_i be the normal to half-plane i, so that (m_i, n_i, t) is an oriented orthonormal basis of $\mathbb{R}^3$. The sector between half-planes i and $i+1$ is equipped with a constant metric $u_{i+1/2} = u_{i+1/2}(\epsilon)$ depending on a small parameter ϵ and we suppose that u is continuous across the half-plane i in the tangential-tangential directions. We suppose that $u_{i+1/2}(0)$ is the canonical Euclidian metric on $\mathbb{R}^3$, and we will be particularly interested in the first derivative of $u_{i+1/2}$ with respect to ϵ at $\epsilon = 0$, which we denote by $u'_{i+1/2}$.

We want to parallell transport a vector around the edge, by a path going once around it and with respect to the metric u. In each sector parallell transport is trivial, but from one sector to another, say from $i-1/2$ to $1+1/2$ we denote by T_i the matrix of the parallell transport in the basis (m_i, n_i, t). It is defined as follows. The oriented unit normal to face i with respect to $u_{i-1/2}$ is denoted k_i^-, that with respect to $u_{i+1/2}$ is denoted k_i^+. The operator T_i maps m_i to m_i and k_i^- to k_i^+ and t to t. We denote by R_i^j the matrix of the identity operator from basis (m_i, n_i, t) to basis (m_j, n_j, t).

Holonomy from sector $i-1/2$ to itself, in the basis (m_i, n_i) is:

$$E_{i-1/2} = R_{i-1}^i T_{i-1} \cdots R_j^{j+1} T_j \cdots R_i^{i+1} T_i. \tag{41}$$

We have that T_i is orthogonal from $u_{i-1/2}$ to $u_{i+1/2}$. Consequently $E_{i-1/2}$ is orthogonal with respect to $u_{i+1/2}$. In the plane orthogonal to t it is a simple rotation. The angle of this rotation does not depend on i and is the deficit angle θ associated with the edge.

Proposition 3.1. *The derivative of the deficit angle θ at $\epsilon = 0$ is given as a sum of jumps:*

$$\theta' = 1/2 \sum_j u'_{j+1/2}(m_j, n_j) - u'_{j-1/2}(m_j, n_j). \tag{42}$$

Proof. Let $\tilde{m}_i$ be the oriented unit normal to t in face i, with respect to $u_{i-1/2}$ or equivalently $u_{i+1/2}$. Let P_i be the matrix of the identity from basis (m_i, n_i, t) to $(\tilde{m}_i, k_i^-, t)$. We have:

$$E_{i-1/2} = P_i^{-1} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} P_i. \tag{43}$$

Differentiating this expression at $\epsilon = 0$ we get:

$$E'_{i-1/2} = \begin{pmatrix} 0 & -\theta' & 0 \\ \theta' & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{44}$$

Differentiating (41) at $\epsilon = 0$ we obtain, since $T_j(0)$ is the identity operator:

$$E'_{i-1/2} = R^i_{i-1} T'_{i-1} R^{i-1}_i + \cdots + R^i_{j+1} T'_{j+1} R^{j+1}_i + \cdots + T'_i. \tag{45}$$

Let J be the canonical skew matrix:

$$J = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{46}$$

It commutes with all R^j_i. We have:

$$\theta' = -1/2 \operatorname{tr}(J E'_{i-1/2}) \tag{47}$$

$$= -1/2 \sum_j \operatorname{tr}(J T'_j). \tag{48}$$

We determine the terms in this sum. Let $M^\pm_j$ be the matrix in (m_j, n_j, t) of the operator sending (m_j, n_j, t) to $(m_j, k^\pm_j, t)$. We have:

$$T_j = M^+_j (M^-_j)^{-1}. \tag{49}$$

Differentiating at $\epsilon = 0$ we obtain:

$$T'_j = (M^+_j)' - (M^-_j)'. \tag{50}$$

Put:

$$M^\pm_j = \begin{pmatrix} 1 & \alpha^\pm_j & 0 \\ 0 & \beta^\pm_j & 0 \\ 0 & \gamma^\pm_j & 1 \end{pmatrix}. \tag{51}$$

Then:

$$\operatorname{tr}(J T'_j) = (\alpha^+_j)' - (\alpha^-_j)'. \tag{52}$$

Differentiating the identity :

$$u_{j+1/2}(m_j, k^+_j) = 0, \tag{53}$$

yields:

$$u'_{j+1/2}(m_j, n_j) + u_{j+1/2}(m_j, (k^+_j)') = 0, \tag{54}$$

from which it follows that:

$$(\alpha^+_j)' = -u'_{j+1/2}(m_j, n_j). \tag{55}$$

Similarly we have:

$$(\alpha^-_j)' = -u'_{j-1/2}(m_i, n_j). \tag{56}$$

Combining this with (48) and (52) gives the proposition. □

We consider now the general case of a simplicial complex in $\mathbb{R}^3$. For a Regge metric $u \in X_h$ its Regge action $\mathcal{R}$ is defined as follows. For any edge i its length is denoted l_i and its deficit angle θ_i.

$$\mathcal{R} = \sum_i \theta_i l_i. \tag{57}$$

Proposition 3.2. *Let Regge metrics u depend smoothly on a parameter ϵ in a neighborhood of 0, where the metric is constant Euclidean. The derivative of u at $\epsilon = 0$ is denoted u'. Then we have:*

$$\mathcal{R}(\epsilon) = 1/4\epsilon^2 \sum_i \theta_i'(0) l_i(0) u'(t_i, t_i) + \mathcal{O}(\epsilon^3). \tag{58}$$

Proof. We have:

$$\theta_i(0) = 0. \tag{59}$$

Consequently:

$$\begin{aligned}
\mathcal{R}(\epsilon) &= \sum_i (\epsilon\theta_i'(0) + 1/2\epsilon^2\theta_i''(0))\ (l_i(0) + \epsilon l_i'(0)) \ + \ \mathcal{O}(\epsilon^3) &(60)\\
&= \epsilon \sum_i \theta_i'(0) l_i(0) \ + \ \epsilon^2 \sum_i (\theta_i'(0) l_i'(0) + 1/2\theta_i''(0) l_i(0)) \ + \ \mathcal{O}(\epsilon^3). &(61)
\end{aligned}$$

It is a remarkable property of Regge calculus, proved in [31], that for any ϵ:

$$\sum_i \theta_i'(\epsilon) l_i(\epsilon) = 0. \tag{62}$$

This handles the first term in (61). Differentiating we get in addition:

$$\sum_i \theta_i''(0) l_i(0) + \theta_i'(0) l_i'(0) = 0. \tag{63}$$

Inserting this in the second term of (61) we get:

$$\mathcal{R}(\epsilon) = 1/2\epsilon^2 \sum_i \theta_i'(0) l_i'(0) + \mathcal{O}(\epsilon^3). \tag{64}$$

Finally consider an edge i and denote its extremities by x_0 and x_1. We have $l_i(0) = |x_1 - x_0|$ and $t_i = (x_1 - x_0)/|x_1 - x_0|$. We can write:

$$\begin{aligned}
l_i(\epsilon) &= (u(\epsilon)(x_1 - x_0, x_1 - x_0))^{1/2} &(65)\\
&= (|x_1 - x_0|^2 + \epsilon u'(x_1 - x_0, x_1 - x_0) + \mathcal{O}(\epsilon^2))^{1/2} &(66)\\
&= l_i(0)(1 + 1/2\epsilon u'(t_i, t_i)) + \mathcal{O}(\epsilon^2), &(67)
\end{aligned}$$

so that:

$$l_i'(0) = 1/2 l_i(0) u'(t_i, t_i). \tag{68}$$

This completes the proof. □

Now insert expression (42) in (58) and compare with the expressions (19) and (20) for the curl T curl operator. We conclude:

Corollary 3.3. *With the above notations we have the expansion:*

$$\mathcal{R}(\epsilon) = 1/8\epsilon^2 \langle \operatorname{curl} \mathrm{T} \operatorname{curl} u', u' \rangle + \mathcal{O}(\epsilon^3). \tag{69}$$

4 Eigenvalue approximation

Let H denote the space of L^2 symmetric matrix fields, equipped with the standard L^2 product denoted $\langle\cdot,\cdot\rangle$. The corresponding norm is denoted $|\cdot|$. We will also use the Sobolev spaces H^s, so that H corresponds to H^0.

Define, on smooth fields:

$$a(u,v) = \langle \operatorname{curl} \mathrm{T} \operatorname{curl} u, v\rangle. \tag{70}$$

We wish to find a large Hilbert-space on which a is continuous.

Recall that $H = \mathrm{L}^2 \otimes \mathbb{S}$. Let $\mathbb{M}$ be the space of 3×3 matrices, and $\mathbb{A}$ the subspace of antisymmetric ones. It is conventient to introduce $Q : \mathbb{M} \to \mathbb{M}$ defined by:

$$Qu = u^{\mathrm{T}} - (\operatorname{tr} u)I, \tag{71}$$

where I is the identity matrix. The operator $\operatorname{curl} Q^{-1} \operatorname{curl}$ is an extension of $\operatorname{curl} \mathrm{T} \operatorname{curl}$ to all matrix fields, which is 0 on antisymmetric ones.

Proposition 4.1. *Let $u \in H$. The following are equivalent:*

- $\operatorname{curl} \mathrm{T} \operatorname{curl} u \in \mathrm{H}^{-1}$*,*
- *There is $v \in \mathrm{L}^2 \otimes \mathbb{A}$ such that $\operatorname{curl}(u+v) \in \mathrm{L}^2 \otimes \mathbb{M}$,*
- *There is $w \in \mathrm{H}^1 \otimes \mathbb{S}$ such that $\operatorname{curl} \mathrm{T} \operatorname{curl} u = \operatorname{curl} \mathrm{T} \operatorname{curl} w$.*

Proof. Suppose the first condition holds, so that $Q^{-1} \operatorname{curl} u \in \mathrm{H}^{-1}$ and also $\operatorname{curl} Q^{-1} \operatorname{curl} u \in \mathrm{H}^{-1}$. Then $Q^{-1} \operatorname{curl} u = v + \operatorname{grad} p$ with $v \in \mathrm{L}^2 \otimes \mathbb{M}$ and $p \in \mathrm{L}^2 \otimes \mathbb{V}$. Now remark that $Q \operatorname{grad} p$ is $\operatorname{curl} \operatorname{skew} p$, so the second condition holds.

Suppose that the second condition holds. Then $u + v = w' + \operatorname{grad} p$ with $w' \in \mathrm{H}^1 \otimes \mathbb{M}$ and $p \in \mathrm{H}^1 \otimes \mathbb{V}$. Let w be the symmetric part of w'. Then:

$$\operatorname{curl} \mathrm{T} \operatorname{curl} u = \operatorname{curl} Q^{-1} \operatorname{curl}(w' + \operatorname{grad} p - v) = \operatorname{curl} \mathrm{T} \operatorname{curl} w. \tag{72}$$

This shows that the third condition holds.

The third condition obviously implies the first. $\square$

Define:

$$X = \{u \in H \ : \ \operatorname{curl} \mathrm{T} \operatorname{curl} u \in \mathrm{H}^{-1}\}. \tag{73}$$

The norm of X will be denoted simply $\|\cdot\|$. From the above proposition it follows that a extends to a continuous bilinear form on X.

Eigenpairs of a are couples $(u,\lambda) \in X \times \mathbb{R}$ such that $u \neq 0$ and:

$$\operatorname{curl} \mathrm{T} \operatorname{curl} u = \lambda u, \tag{74}$$

which can be rephrased as:

$$\forall v \in X \quad a(u,v) = \lambda\langle u, v\rangle. \tag{75}$$

We will be interested in non-zero eigenvalues, the eigenvalue zero correspond to the kernel of a, which consists of deformation tensors and a finite dimensional cohomology space.

Define:

$$W = \{u \in H \ : \ \mathrm{curl}\,\mathrm{T}\,\mathrm{curl}\, u = 0\}, \tag{76}$$

and:

$$V = \{u \in X \ : \ \forall w \in W \quad \langle u, w\rangle = 0\}, \tag{77}$$

so that we have a decompostion:

$$X = V \oplus W. \tag{78}$$

We also define $\overline{V}$ to be the orthogonal in H:

$$\overline{V} = \{u \in H \ : \ \forall w \in W \quad \langle u, w\rangle = 0\}. \tag{79}$$

Then $\overline{V}$ is indeed the closure of V in H. We have:

$$H = \overline{V} \oplus W. \tag{80}$$

We let $P : H \to H$ be the projector with range $\overline{V}$ and kernel W. It induces continuous projections in X with range V.

Since $\mathrm{curl}\,\mathrm{T}\,\mathrm{curl} : X \to \mathrm{H}^{-1}$ has closed range we have:

$$\inf_{u\in V} \sup_{v \in V} \frac{|a(u,v)|}{\|u\| \ \|v\|} > 0. \tag{81}$$

Define a map $K : H \to V$ by imposing:

$$\forall v \in V \quad a(Ku, v) = \langle u, v\rangle. \tag{82}$$

The induced map $K : H \to H$ is compact since it is continuous from $H \to V$ and the injection $V \to H$ is compact. It is also selfadjoint. Therefore H has an orthonormal basis consisting of eigenvectors. Moreover λ is a non-zero eigenvalue of K iff $1/\lambda$ is a non-zero eigenvalue of a. Notice that K has both positive and negative eigenvalues.

We now turn to the discrete eigenvalue problem.

Unfortunately X_h is not a subspace of X. For $\delta \in]-1/2, 1/2[$ define X^δ by:

$$X^\delta = \{u \in H \ : \ \mathrm{curl}\,\mathrm{T}\,\mathrm{curl}\, u \in \mathrm{H}^{-1+\delta}\}. \tag{83}$$

The corresponding norm is denoted $\| \cdot \|_\delta$. Thus $X = X^0$. Then a extends to a continuous bilinear form on $X^\delta \times X^{-\delta}$. We have that W is a closed subspace of X^δ for any δ. Let V^δ be its orthogonal in X^δ:

$$V^\delta = \{u \in X^\delta \ : \ \forall w \in W \quad \langle u, w\rangle = 0\}. \tag{84}$$

We have:

$$X^\delta = V^\delta \oplus W, \tag{85}$$

and:

$$\inf_{u\in V^\delta} \sup_{v \in V^{-\delta}} \frac{|a(u,v)|}{\|u\|_\delta \ \|v\|_{-\delta}} > 0. \tag{86}$$

The projector P induces continuous projections in X^δ with range V^δ.

Notice that $X_h \subset X^\delta$ for $\delta < 0$ but not for $\delta = 0$.

For each h we look for pairs $(u, \lambda) \in X_h \times \mathbb{R}$ with $u \neq 0$ such that:

$$\forall u' \in X_h \quad a(u, u') = \lambda \langle u, u' \rangle. \tag{87}$$

The 0 eigenvalues correspond to the range of def : $X_h^0 \to X_h^1$ and cohomology classes. We will henceforth be interested in non-zero eigenvalues.

Define decompositions of X_h as follows. First put:

$$W_h = \{u \in X_h \ : \ \forall v \in X_h \quad a(u, v) = 0\}. \tag{88}$$

Notice that in fact we have:

$$W_h = X_h \cap W. \tag{89}$$

Next define:

$$V_h = \{u \in X_h \ : \ \forall w \in W_h \quad \langle u, w \rangle = 0\}. \tag{90}$$

We then have decompositions:

$$X_h = V_h \oplus W_h. \tag{91}$$

Notice that V_h is no a subspace of $\overline{V}$. We procced to prove that the spaces are close in a sense:

Proposition 4.2. *Fix $\delta < 0$. There is a sequence ϵ_h converging to 0 such that for all h and all $v_h \in V_h$:*

$$|v_h - Pv_h| \leq 2\epsilon_h \|v_h\|_\delta. \tag{92}$$

Proof. By composing I_h with a regularization operator R_h commuting with $\operatorname{curl} \mathrm{T} \operatorname{curl}$ we get operators $I_h R_h : X \to X_h$ which are arbitrarily close to the identity on X_h, stable in H and map W to W. The restriction $I_h R_h : X_h \to X_h$ is invertible. Compose $I_h R_h$ to the left with this inverse. We get projections $\Pi_h : X \to X_h$ which are uniformly bounded in H and send W to W_h. The regularization operator R_h is essentially regularization by convolution with a function $\phi_{\epsilon h} = (\epsilon h)^{-3} \phi((\epsilon h)^{-1} \cdot)$, where the parameter ϵ is fixed but small. This type of construction is detailed for finite element exterior calculus in [3] and [19]. See also [36] and [18] for earlier variants.

We have by the approximation property of (X_h) in H and the uniform boundedness of the projectors Π_h:

$$\forall u \in H \quad \Pi_h u \to u \text{ in } H. \tag{93}$$

Since the injection $V^\delta \to H$ is compact, there is a sequence ϵ_h converging to 0 as $h \to 0$ such that:

$$\forall u \in V^\delta \ \forall h \quad |u - \Pi_h u| \leq \epsilon_h \|u\|_\delta. \tag{94}$$

Choose v_h in V_h.Write:

$$|v_h - Pv_h| \leq |v_h - \Pi_h P v_h| + |Pv_h - \Pi_h P v_h|. \tag{95}$$

We have:

$$|Pv_h - \Pi_h P v_h| \leq \epsilon_h \|Pv_h\|_\delta \leq \epsilon_h \|v_h\|_\delta. \tag{96}$$

Remark that $v_h - Pv_h \in W$ so, by our hypothesis on Π_h, we have:

$$v_h - \Pi_h P v_h = \Pi_h (v_h - P v_h) \in W_h. \tag{97}$$

Therefore v_h and Pv_h are both orthogonal to $v_h - \Pi_h P v_h$:

$$|v_h - \Pi_h P v_h|^2 = \langle v_h - \Pi_h P v_h, P v_h - \Pi_h P v_h \rangle, \tag{98}$$

$$\leq |v_h - \Pi_h P v_h| \; |P v_h - \Pi_h P v_h|, \tag{99}$$

so that:

$$|v_h - \Pi_h P v_h| \leq |P v_h - \Pi_h P v_h|. \tag{100}$$

Thus we get:

$$|v_h - P v_h| \leq 2|P v_h - \Pi_h P v_h| \leq 2\epsilon_h \|v_h\|_\delta, \tag{101}$$

which concludes the proof. □

The above estimate immediately gives:

$$\|v_h - P v_h\|_\delta \leq 2\epsilon_h \|v_h\|_\delta. \tag{102}$$

which shoes that the gap from V_h to V^δ in X^δ goes to 0.

Next we prove a rather weak inf-sup condition:

Proposition 4.3. *Fix $\delta < 0$. There is C such that for all h:*

$$\forall u \in X_h \quad \|Pu\|_\delta \leq C \sup_{v \in X_h} \frac{|a(u,v)|}{|v|}. \tag{103}$$

Proof. We first evaluate the norm of the restriction operator from a tetrahedron $T \in \mathcal{T}_h$ to an edge i in norms $\mathrm{H}^{1-\delta}(T) \to \mathrm{L}^2(i)$.

We let $\hat{T}$ be a reference simplex of diameter 1. We suppose that $x \mapsto hx$ maps $\hat{T} \to T$ and the edge $\hat{i} \to i$. For u on T we define $\hat{u}$ on $\hat{T}$ by $\hat{u}(x) = u(hx)$. We have:

$$\int_i |u|^2 = h \int_{\hat{i}} |\hat{u}|^2 \tag{104}$$

$$= Ch \|\hat{u}\|^2_{\mathrm{H}^{1-\delta}(\hat{T})} \tag{105}$$

$$\leq Ch^{-2} \|\hat{u}\|^2_{\mathrm{H}^{1-\delta}(T)}. \tag{106}$$

It follows that we have an estimate:

$$(\sum_i \int_i |u|^2)^{1/2} \leq Ch^{-1} \|u\|_{\mathrm{H}^{1-\delta}}. \tag{107}$$

Suppose now that $u \in X_h^2$. We can write:

$$u = \sum_i u_i t_i t_i^{\mathrm{T}} \delta_i, \tag{108}$$

where for each edge i, $u_i \in \mathbb{R}$. We have:

$$\|u\|_{-1+\delta} = \sup_{v \in \mathrm{H}^{1-\delta}} \frac{|\sum_i \int_i u_i t_i t_i^{\mathrm{T}} \cdot v|}{\|v\|_{\mathrm{H}^{1-\delta}}} \tag{109}$$

$$\leq Ch^{-1} (\sum_i |u_i|^2 h)^{1/2} \tag{110}$$

$$\leq h^{-1/2} (\sum_i |u_i|^2). \tag{111}$$

Choose $u \in X_h$. Define numbers u_i by:

$$\operatorname{curl} \mathrm{T} \operatorname{curl} u = \sum_i [u]_i t_i t_i^{\mathrm{T}} \delta_i, \tag{112}$$

Choose v to be the element of X_h with degrees of freedom $[u]_i$. We have:

$$|\langle \operatorname{curl} \mathrm{T} \operatorname{curl} u, v\rangle|/|v| \geq 1/C h^{-1/2} (\sum_i [u]_i^2)^{1/2} \tag{113}$$

$$\geq 1/C \| \operatorname{curl} \mathrm{T} \operatorname{curl} u\|_{-1+\delta} \tag{114}$$

$$\geq 1/C \|Pu\|_\delta. \tag{115}$$

This completes the proof. □

Corollary 4.4. *Fix $\delta < 0$. There is C such that for all h:*

$$\inf_{u \in V_h} \sup_{v \in V_h} \frac{|a(u,v)|}{|u| \|v\|_\delta} = \inf_{u \in V_h} \sup_{v \in V_h} \frac{|a(u,v)|}{\|u\|_\delta |v|} \geq 1/C. \tag{116}$$

Proof. The equality reflects that a map and its adjoint have the same norm. The inequality follows from the preceding theorem using that for $u_h \in V_h$:

$$\|u\|_\delta \leq (1 - 2\epsilon_h)^{-1} \|Pu\|_\delta, \tag{117}$$

and that the decomposition $X_h = V_h \oplus W_h$ is orthogonal in H. □

We introduce the discrete analogue of K, which is the map $K_h : H \to V_h$ taking any $u \in H$ to the element $K_h u \in V_h$ such that:

$$\forall u' \in V_h \quad a(K_h u, u') = \langle u, u' \rangle. \tag{118}$$

The operator K_h has finite rank and is symmetric. Remark that λ is a non-zero discrete eigenvalue (of a on X_h) iff $1/\lambda$ is a non-zero eigenvalue of K_h. The corresponding eigenspaces are the same. In order to relate the discrete eigenvalues of a to the continuous ones, we shall prove:

$$\|K - K_h\|_{H \to H} \to 0. \tag{119}$$

Still imposing $\delta < 0$, let $P_h : X^{-\delta} \to V_h$ be the projector defined by, for all $u \in X^{-\delta}$, $P_h u$ is the element of V_h such that:

$$\forall v \in V_h \quad a(P_h u, v) = a(u, v). \tag{120}$$

This equation then holds for all $v \in X_h$. Notice that we have, for all $u \in H$ all h and all $v \in V_h$:

$$a(K_h u, v) = \langle u, v \rangle = a(Ku, v) = a(P_h K u, v). \tag{121}$$

Therefore:

$$K_h = P_h K. \tag{122}$$

We will need a lemma.

Lemma 4.5. *Let $u_h = v_h + w_h$ with $v_h \in V_h$ and $w_h \in W_h$. Suppose that $u_h \to u$ in H with $v \in \overline{V}$ and $w \in W$. Then $v_h \to v$ and $w_h \to w$ in H.*

Proof. We have:

$$0 \leftarrow |u - u_h|^2 = |v - v_h|^2 + |w - w_h|^2 - 2v_h \cdot w. \tag{123}$$

Moreover:

$$|v_h|^2 + |w_h|^2 = |u_h|^2 \leq C, \tag{124}$$

and:

$$|v_h \cdot w| = |v_h \cdot (w - \Pi_h w)| \leq |v_h| \, |w - \Pi_h w|, \tag{125}$$

where Π_h is as in the proof of Proposition 4.2. Combining these three estimates gives the lemma. □

Proposition 4.6. *The maps P_h are uniformly bounded from $X^{-\delta}$ to H. And for any $u \in H$, we have convergence $P_h K u \to K u$ in H.*

Proof. Uniform boundedness follows from the inf-sup condition of Corollary 4.4.

Moreover we have, for $u \in H$ and $v \in X_h$, using the computation carried out in the proof of Proposition 2.4 and regularization operators R_h as in Proposition 4.2:

$$a(P_h K u - I_h R_h K u, v) = a(K u - R_h K u, v). \tag{126}$$

Hence:

$$\sup_{v \in X_h} |a(P_h K u - I_h R_h K u, v)| / \|v\|_\delta \leq \|K u - R_h K u\|_{-\delta} \to 0. \tag{127}$$

We write $I_h R_h K u = v_h + w_h$ with $v_h \in V_h$ and $w_h \in W$. We have from (127) that:

$$|P_h K u - v_h| \to 0. \tag{128}$$

On the other hand, from Lemma 4.5 we have:

$$|K u - v_h| \to 0. \tag{129}$$

This completes the proof. □

Corollary 4.7. *We have the convergence $K_h \to K$ in norm $H \to H$.*

Proof. In view of the preceding proposition it is enough to check that the map $K : H \to X^{-\delta}$ is compact. Suppose (u_n) converges weakly to 0 in H. Then we have by (86):

$$\|K u_n\|_{-\delta} \leq \sup_{v \in V^\delta} |a(K u_n, v)| / \|v\|_\delta \tag{130}$$

$$\leq \sup_{v \in V^\delta} |\langle u_n, v \rangle| / \|v\|_\delta, \tag{131}$$

This converges to 0 since the injection $V^\delta \to H$ is compact. □

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